

A MULTIVARIATE RATE EQUATION FOR VARIABLE-INTERVAL PERFORMANCE

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A value-like parameter is introduced into a rate equation for describing variable-interval performance. The equation, derived solely from formal considerations, expresses rate of responding as a joint function of rate of reinforcement and "reinforcer power." Preliminary tests of the rate equation show that it handles univariate data as well as Herrnstein's hyperbola. In addition, a form of Herrnstein's hyperbola can be derived from the equation, and it predicts forms of matching in concurrent situations. For the multivariate case, reinforcer values scaled in concurrent situations where matching is assumed to hold are taken as determinations of reinforcer power. The multivariate rate equation is fitted to an appropriate set of data and found to provide a good description of variable-interval performance when both rate and power of reinforcement are varied. Rate and power measures completely describe reinforcement. The effects of their joint variation are not predicted and cannot be described by Herrnstein's equation.

Key words: value, mathematics, variable-interval schedules, linear systems, reinforcement rate, response rate, matching, law of effect

For responding maintained by variable-interval (VI) reinforcement, Herrnstein's hyperbola,

$$R = \frac{kr}{r + r_0}, \quad (1)$$

expresses rate of responding (R) as a function of one independent variable, viz., rate of reinforcement (r). The equation can also be made to accommodate amount or immediacy of reinforcement as an independent variable by directly substituting one or the other in place of r (deVilliers & Herrnstein, 1976). The empirical accuracy of Herrnstein's equation (deVilliers & Herrnstein, 1976) indicates that the relationship between response rate and each of these reinforcement parameters is hyperbolic. However, the equation does not specify how the parameters combine to determine response rate.

The problem of how various parameters of reinforcement jointly govern behavior has

been discussed by a number of authors in the context of relative response rate matching (Baum, 1973; Baum & Rachlin, 1969; Killeen, 1972; Rachlin, 1971). According to Killeen (1972), who has provided the most general treatment, reinforcer "value" (V) may be defined by a multiplicative combination of unspecified functions of rate (r), amount (a), immediacy (i), and other presumably important parameters (x), of reinforcement:

$$V = f_1(r) f_2(a) f_3(i) f_4(x). \quad (2)$$

Baum & Rachlin (1969) argue that matching occurs with respect to value:

$$\frac{T_1}{T_2} = \frac{V_1}{V_2},$$

where T_1 is the time allocated to one response alternative, V_1 is the value of reinforcement accruing to that responding, and T_2 and V_2 are corresponding quantities for a second response alternative. Determining the forms of the various functions in Equation 2 is presumably an empirical matter, as is the specification of the additional reinforcement parameters represented by x .

The idea of reinforcer value has been discussed only cursorily in the context of single-alternative responding (see deVilliers & Herrnstein, 1976). Yet absolute response rate varies as a function of at least three different param-

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eters of reinforcement, and Herrnstein's hyperbola can account for these variations only by taking the parameters one at a time. In the present paper, we introduce a value-like parameter into a rate equation for describing VI performance. In contrast to the empirical approach characteristic of matching, however, we have taken an a priori mathematical approach to the problem of value.² A mathematical description of reinforcement and responding on VI schedules, and the derivation of a multivariate rate equation for VI responding, will be presented first. The remainder of the paper will discuss the accuracy and utility of the multivariate rate equation.

THE MATHEMATICS OF VI PERFORMANCE

Simple time dependent schedules of reinforcement permit a straightforward description of behavior. Simple time dependence characterizes those schedules where both reinforcement and behavior are expressed only as functions of time. The schedule and the resulting behavior may be expressed as two separate functions. The function describing the schedule will be designated $R(t)$, i.e., reinforcement as a function of time; and the function describing the resulting behavior will be designated $B(t)$, i.e., behavior, or operant responding, as a function of time. The pair of functions, $R(t)$ and $B(t)$, constitute the environment and the resulting behavior of an organism. The organism is characterized by a transfer function between $R(t)$ and $B(t)$. The transfer function is not an algebraic function but is more in the nature of an operator since its argument must be the function $R(t)$. For the general case, a transfer function is usually designated G . If both $R(t)$ and $B(t)$ can be written explicitly, the task of describing behavior is reduced to the problem of finding an appropriate transfer function.

For the case of an organism responding on a simple time dependent schedule of reinforcement, the observer sees the behavior of

a one-port black box (one input and one output). The term "black box" means that the analysis is concerned with the transfer function only, not with the actual mechanisms by which the transfer function occurs in the organism. The analysis can be generalized to the case of an organism responding to more than one simple time-dependent schedule. In the most general case, the observer sees the behavior of an m-input, n-output black box. The term "network" is often used to describe a many-ported black box. Although the calculations involved for a more complex network are likely to be difficult, it is important to note that many types of behavior may be analyzed as networks.

A VI schedule presents an organism with a single time-dependent input, and in response the organism produces a single time-dependent output (cf. Staddon, 1964). Reinforcements and responses on VI schedules occur with random-length intervals between them. The mean values of these intervals are time-averaged constants. A VI transfer function can relate only these averaged input and output properties. What is shown in the following pages is a first calculation of a transfer function between the time-averaged properties of VI reinforcement and responding.

Let reinforcements and responses occur at a sequence of times t_i and t^*_i ("t star sub-i") respectively. That is, at $t = t_i$ a reinforcement occurs, and at $t = t^*_i$ a response is executed. The t_i 's and t^*_i 's must satisfy two conditions. First, the specific points at which the t_i 's and t^*_i 's occur must be random.³ Second, a constant time-averaged rate of reinforcement must produce a constant time-averaged rate of responding. These conditions are required by the empirical nature of VI schedules and the performance they generate (see, e.g., Catania & Reynolds, 1968). A function having these properties, and involving the t_i 's must be written for the reinforcement signal, and a similar function involving the t^*_i 's must be written for the response output.

Assumptions

The selection of an exact functional form for $R(t)$ and $B(t)$ is based on a number of as-

²Anderson's (1974) information integration theory is another mathematical approach to the problem of value. Farley and Fantino (1978) have applied Anderson's stochastic model to concurrent-chains procedures with good results. However, they do not discuss single-alternative responding in detail and it is not clear how Anderson's model could be used to describe responding in the simpler case.

³That is, knowledge of any subsequence of the points t_i or t^*_i , for the VI case, carries no information about where the points t_i or t^*_i occur in any other disjoint subsequence.

sumptions. These assumptions reflect how we choose to describe reinforcement and responding. Clearly, the functional form we have selected for $R(t)$ and $B(t)$ is not unique. However, other forms must satisfy the conditions already expressed for the t_i 's and the t^*_i 's.

We require four assumptions in order to write $R(t)$ and $B(t)$ explicitly. First, the organism is assumed to "change state" whenever it is reinforced and whenever it executes an instrumental response. The size of the change is given by the value-like parameter, A . The symbol A_R (reinforcer amplitude) will represent the change due to reinforcement; the symbol A_B (response amplitude) will represent the change due to responding.

This change of state assumption means that we conceive of the organism as existing in one of three states at a given time, t . When it is executing the instrumental response, we say that it is in the "response state"; when it is eating, we say that it is in the "reinforcement state"; and when it is in neither of these states we say that it is in the "zero state." The critical consequence of this assumption is that the value of $R(t)$ and $B(t)$ changes at each t_i and t^*_i . In other words, the change of state assumption produces functions for $R(t)$ and $B(t)$ that are not identically zero (or constant) for all t .

It is important to emphasize that this assumption is critical from a mathematical point of view. It implies the existence of nonzero (or nonvanishing) time derivatives of $R(t)$ and $B(t)$ at the reinforcement points (the t_i 's), and at the response points (the t^*_i 's). This means that $R(t)$ and $B(t)$ contain information about where reinforcements and responses occur. The nonmathematical conceptualization of the assumption in terms of "states" of the organism, on the other hand, is not crucial. One could just as well say that there is "something different" about the points where reinforcements and responses occur, or that the organism's environment changes at the reinforcement points and its behavior changes at the response points. We have chosen the more elaborate "change of state" conceptualization because it seems to make the mathematics clearer.

Our second assumption is that each individual response or reinforcement is identical to all other like events. For food reinforcement, for example, the i th feeding is assumed to affect the organism in exactly the same way

as the $(i + 1)$ th feeding. Similarly, for a given type of responding, the i th response is assumed to be identical to the $(i + 1)$ th response. This assumption places an important constraint on responding since the amplitude, A_B , of a given response is assumed to be locked, i.e., invariant, across different types (and hence amplitudes) of reinforcement. On the other hand, reinforcer amplitude, A_R , is a free parameter that may be varied experimentally.

Our third assumption is that the length of time during which the organism is in either the reinforcement or the response state is finite, although it may be very small. A finite "dead time" per reinforcement and response implies an upper limit on the rates of reinforcement and responding.

And finally, the exact form of the transition between states is assumed to be a jump discontinuity. In other words, the functions $R(t)$ and $B(t)$ are assumed to appear as a series of rectangular pulses. A discontinuous transition was selected because it is the easiest to write. Clearly, there is no a priori reason to select any other form (e.g., gaussian, dirac delta, or rational).

$R(t)$ and $B(t)$

To give a clear picture of how $R(t)$ is written, an expression for a single reinforcement will be written first. Since the form chosen is a rectangular pulse as a function of time, a single reinforcement is given by

$$r(t) = \begin{cases} A_R & t_i < t < t_i + w \\ 0 & t < t_i \text{ or } t > t_i + w \end{cases} \quad (3)$$

where t_i is the initial time of the reinforcement, and w is the length of time during which the organism is in the changed state. The inequalities on the right hand side of Equation 3 specify the times for which the function $r(t)$ assumes each of its two ordinate values. When t is between t_i and $t_i + w$ (the top inequality), the value of the function is A_R . When t is less than t_i or greater than $t_i + w$ (the bottom inequality), the value of the function is zero. The function must equal either A_R or zero at any time t (except at the points of discontinuity, t_i and $t_i + w$, where the function is undefined), and cannot assume any other value. The lower-case r on the left hand side of the equation simply indicates that a single reinforcement pulse is being considered. A plot of Equation 3 is shown in Figure 1. The function is identically zero until t_i

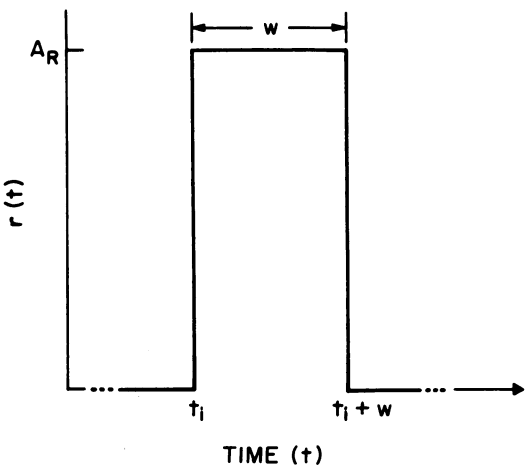


Fig. 1. A single reinforcement pulse or spike. A_R is the amplitude of the pulse. For $t_i < t < t_i + w$, $r(t) = A_R$.

after which point it becomes A_R . After an additional period of time w , the function returns to zero.

For a VI input, the complete reinforcement function, $R(t)$, appears as a randomly distributed row of spikes, or pulses, of the form given by Equation 3. That is,

$$R(t) = \begin{cases} A_R & t_i < t < t_i + w \\ 0 & t_i + w < t < t_{i+1} \end{cases} \tag{4}$$

where the t_i 's occur randomly along the t -axis for $t > t_0$. The point t_0 is the beginning of the session. Clearly, $R(t)$ is identically zero for all $t < t_0$. The inequalities on the right hand side of Equation 4 specify the times for which the function $R(t)$ assumes each of its two ordinate values. During a reinforcement pulse (the top inequality), the value of the function is A_R . Between reinforcement pulses (the bottom inequality), the value of the function is zero.

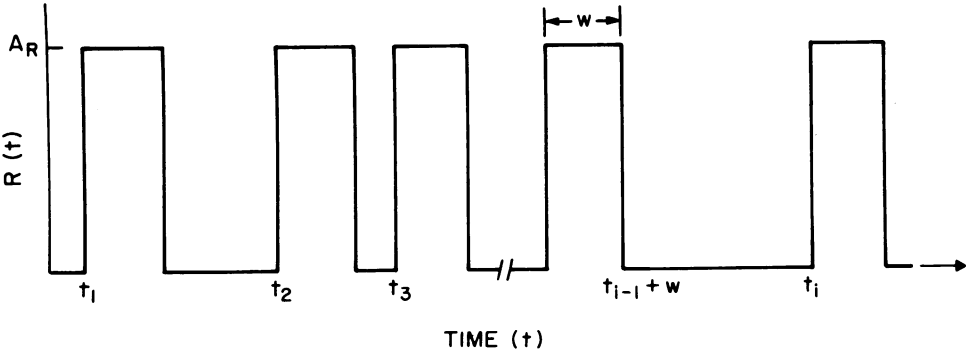


Fig. 2. A train of reinforcement pulses. Pulses occur at $t = t_1, t_2, t_3, t_{i-1}, t_i$. The amplitude of each pulse is A_R . The duration of each pulse is w .

A section of $R(t)$ is shown in Figure 2. Notice that the i^{th} interreinforcement interval is given by $t_i - (t_{i-1} + w)$, which is the difference between the initial or "up" time of a given reinforcement pulse, and the final or "down" time of the preceding pulse. The average interreinforcement interval (i.e., the VI value) will be designated \bar{t}_I . The inverse of the average interreinforcement interval is the average rate of reinforcement, which will be designated R_{in} ("rate in"). The inverse relationship between \bar{t}_I and R_{in} can be stated compactly as

$$1/\bar{t}_I = R_{in} \tag{5}$$

Since the average rate of reinforcement on a VI schedule is a constant, \bar{t}_I must also be a constant. This constraint is expressed in the following limit:

$$\bar{t}_I = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n t_i - (t_{i-1} + w)}{n} \tag{6}$$

The expression on the right is simply the sum of the n interreinforcement intervals divided by n .

With Equation 6 giving the condition that the t_i 's of Equation 4 must satisfy, the reinforcement function, $R(t)$, is complete. Notice that the quantities of importance in connection with $R(t)$ are A_R , the amplitude of the reinforcement pulse; w , the width of the pulse; and \bar{t}_I , the inverse of which is the average rate of reinforcement, R_{in} .

The $R(t)$ axis is orthogonal to time and, for conceptual clarity, may be thought of as a value-like dimension. Thus a larger spike amplitude in Figure 2 would correspond to a more "highly valued" reinforcer. In line with

the change of state assumption, one may conceive of the organism as jumping (or being boosted) to the reinforcement state at each t_i and returning (or falling back down) to the zero state w units of time later. The greater the value of the reinforcer, the bigger the $R(t)$ -axis jump.

The development of the response function, $B(t)$, is exactly analogous to that of the reinforcement function and will not be presented in detail. Briefly,

$$B(t) = \begin{cases} A_B & t^*_i < t < t^*_i + w^* \\ 0 & t^*_i + w^* < t < t^*_{i+1} \end{cases} \quad (7)$$

where A_B , the amplitude of the response pulse, need not equal A_R , and w^* the width of the response pulse, need not equal w . The inequalities specify the times for which the function $B(t)$ assumes each of its two ordinate values. During a response pulse (the top inequality), the value of the function is A_B . Between response pulses (the bottom inequality), the value of the function is zero. Also, if \bar{t}^*_I is the average interresponse time, and R_{out} ("rate out") the average response rate, then

$$1/\bar{t}^*_I = R_{out}, \quad (8)$$

and

$$\bar{t}^*_I = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n t^*_i - (t^*_{i-1} + w^*)}{n}. \quad (9)$$

Equation 8 states that the inverse of the average interresponse time is equal to the average response rate. The expression on the right hand side of Equation 9 is the sum of the n interresponse times divided by n . This equation states that the average interresponse time (or average response rate) is constant on a VI schedule. The $B(t)$ axis is also orthogonal to time and may likewise be thought of as a value-like dimension.

Equations 4 through 9 are not very remarkable. If Figure 2 is considered an enlarged event record of VI reinforcement or responding, Equations 4 and 7 are simply mathematical descriptions of the lines drawn by the event pen. The remaining four equations define two important properties of the VI event record, namely, that the average interreinforcement and interresponse intervals are invariant when large sections of the record are compared, and that the inverses of these average intervals (viz., the average re-

inforcement and response rates) are necessarily invariant in the same sense. The difference between an event record and Figure 2 is that in the former the y-axis is arbitrary and behaviorally meaningless. In Figure 2 ordinates correspond to value-like quantities.

Linear Systems and the Laplace Method

Since expressions for $R(t)$ and $B(t)$ have been written the calculation of a transfer function on $R(t)$ is now possible. The method used in the calculation is a modification of the Laplace transform method of linear system analysis. The Laplace method is common in electrical engineering and other areas of applied science, but since it is undoubtedly unfamiliar to most behavioral scientists, some general features of the method will be presented in this section.

The only requirement for applying the Laplace method is that the system under study be linear. A system is linear if it can be described by a linear differential equation. A differential equation is linear if the dependent variable and its derivatives occur only in the first degree. Many linear differential equations that occur in the applied sciences are difficult to solve. The advantage of the Laplace transform method is that it allows one to calculate the output of a linear system for any given input, without having to solve the differential equation that describes the system.

The Laplace method is typically applied in three steps. First, the differential equation that describes the system is written. Second, the Laplace transforms of both sides of the equation are calculated. The Laplace transform, $F(s)$, of a function, $F(t)$, is defined as

$$F(s) = \int_0^{\infty} F(t)e^{-st} dt,$$

Transforming both sides of a linear differential equation isolates the Laplace transform, $F_i(s)$, of the input, $F_i(t)$, on one side of the equation, and the Laplace transform, $F_o(s)$, of the output, $F_o(t)$, on the other side. (The subscript "o" indicates output; the subscript "i" indicates input.) Notice that the original (untransformed) expressions are functions of time, t , and that the transformed expressions are functions of the Laplace variate, s .

In the third step of the method, the transfer function, $G(s)$, is calculated from the following definition:

$$\frac{F_o(s)}{F_i(s)} = G(s).$$

The transfer function is the ratio of the transformed output to the transformed input. This equation can also be written

$$F_o(s) = G(s)F_i(s).$$

The transfer function, $G(s)$, is said to characterize the system. It is independent of any externally applied input and is expressed in terms of the system's components which may be resistors and capacitors in the case of an electrical circuit, or masses and force constants in the case of a mechanical system. The input or "forcing function", $F_i(t)$, is considered to drive the system, $G(s)$, which responds by producing the output signal, $F_o(t)$. Once $G(s)$ is known, the output that will be produced by any given input can be calculated from the definition of the transfer function,

$$F_{o'}(s) = G(s) F_{i'}(s),$$

where the primes indicate new functions. The new output, $F_{o'}(t)$, is found by taking the inverse transform of $F_{o'}(s)$. The application of the Laplace transform method is especially convenient because extensive tables of Laplace transforms are available. A complete discussion of the mathematical theory of linear systems can be found in Brown (1961), and Pipes and Harvill (1970). Aseltine (1958) has written a particularly readable account of the theory and of the use of transform methods in the analysis of linear systems. A simple example of the Laplace method is given in Appendix D.

Because a differential equation describing an organism as a linear system has not been written, several modifications of the Laplace method are required when it is applied to $R(t)$ and $B(t)$. First, the organism is *assumed* to be a linear system. That is, a linear differential equation describing the organism is assumed to exist. It is important to note that this assumption does not imply a linear relationship between specific properties of the reinforcement and response signals. For example, it does not imply that response and reinforcement rates are linearly related. Second, the transfer function, $G(s)$ cannot be calculated in the usual fashion (i.e., from the differential equation; cf. the example in Appendix D) but must be calculated from the ratio of the transformed signals:

$$\frac{B(s)}{R(s)}.$$

This means that $G(s)$ will not be explicitly defined in terms of the organism's characteristics. As a consequence, it will be impossible to obtain a new output, $B_{new}(t)$, from a new input in the usual fashion, i.e., by multiplying $R_{new}(s)$ by $G(s)$ and consulting a table of inverse Laplace transforms. However, $G(s)$ may be treated as a parameter by fixing s at 1 and writing the transfer function as

$$\frac{B(s=1)}{R(s=1)} = \gamma$$

or

$$B(s=1) = \gamma R(s=1),$$

where $\gamma = G(s=1)$, and is a scalar constant for a given system. This expression relates the transformed input signal to the transformed output signal. It is evaluated at $s=1$ for convenience, since the expression must be true for all values of s where the real part of $s > 0$. Although this expression does not relate the original input and output functions, $R(t)$ and $B(t)$, it permits the calculation of relationships between certain properties of those functions, as will be shown in the next section.

It is evident that all the benefits of the Laplace method are not realized in this behavioral application. If $G(s)$ were expressed explicitly, i.e., if a differential equation describing the system were known, then a $B(t)$ could be determined for any $R(t)$. With $G(s)$ fixed as a parameter, however, the final equation holds only for the specific $R(t)$ that has entered into the calculation. In the present case, the final equation holds only for VI schedules.

Even though the full benefits of the Laplace method are not realized here, the behavioral application shows the power of the method in generating mathematical descriptions of empirical phenomena, even when one's understanding of the systems that produce them is inadequate, i.e., even when the exact forms of the differential equations that describe the systems are not known.

A Transfer Function on $R(t)$:

The Multivariate Rate Equation

The Laplace method is applied to $R(t)$ and $B(t)$ in a straightforward manner. The first step in the calculation is finding the Laplace

transforms of the two functions. The transform of $R(t)$ is defined as

$$R(s) = \int_0^{\infty} R(t)e^{-st} dt.$$

Substituting $R(t)$ as given by Equation 4 into this expression produces

$$R(s) = \sum_{i=1}^{\infty} \int_{t_i}^{t_i+w} A_R e^{-st} dt.$$

Notice that $R(t)$ contributes to the integral (which can be thought of as the area under the function) only when t is between t_i and $t_i + w$. This is because the function assumes its nonzero value (viz., A_R) only during a reinforcement pulse. The integral in the above expression is, in effect, the Laplace transform of a single reinforcement pulse. The complete transform is obtained by summing over all the pulses, as indicated by the infinite sum.

Evaluating the integral gives the transformed version of $R(t)$:

$$R(s) = (A_R/s)(1 - e^{-sw}) \sum_{i=1}^{\infty} e^{-st_i}. \quad (10)$$

A detailed evaluation of the integral is given in Appendix A.

By analogy, the Laplace transform of $B(t)$ is

$$B(s) = (A_B/s)(1 - e^{-sw}) \sum_{i=1}^{\infty} e^{-st_i}. \quad (11)$$

The transfer function, $G(s)$, is calculated by taking the ratio of transformed output to transformed input:

$$\frac{B(s)}{R(s)} = G(s). \quad (12)$$

Substituting Equations 10 and 11 into Equation 12, solving for $B(s)$, setting $s = 1$, and letting $\gamma = G(s=1)$ gives

$$A_B(1 - e^{-sw}) \sum_{i=1}^{\infty} e^{-t_i} = \gamma A_R(1 - e^{-sw}) \sum_{i=1}^{\infty} e^{-t_i}. \quad (13)$$

The application of the Laplace method is now complete. It remains only to write Equation 13 in a useful form. Since we are interested in the relationship between rates of responding and rates of reinforcement it will be helpful to express t_i and t_i^* in terms of \bar{t}_I and \bar{t}_I^* which are the inverses of the VI reinforcement and response rates. Notice that t_i and t_i^* appear in the infinite sums on either

side of Equation 13. To express t_i in terms of \bar{t}_I , recall from Equation 6 that

$$\bar{t}_I \cong \frac{\sum_{i=1}^n t_i - (t_{i-1} + w)}{n}.$$

This equation states that the sum of the n interreinforcement intervals (numerator on the right) divided by n gives the average interreinforcement interval. Since w is a constant,

$$\bar{t}_I \cong \frac{\sum_{i=1}^n (t_i - t_{i-1})}{n} - w,$$

and

$$\bar{t}_I + w \cong \frac{\sum_{i=1}^n (t_i - t_{i-1})}{n}. \quad (14)$$

Notice that the numerator on the right is the sum of the differences between adjacent t_i 's. If one sums to a t_i other than the n^{th} t_i (the latter sum is indicated in Equation 14), the sum is equal to that t_i . In other words

$$t_i = \sum_{j=1}^i (t_j - t_{j-1}),$$

where the new index, j , is introduced for clarity.⁴ Summing to i in Equation 14, i.e., letting $n = i$ and $i = j$, produces

$$\bar{t}_I + w \cong \frac{\sum_{j=1}^i (t_j - t_{j-1})}{i}.$$

This expression is true provided i is sufficiently large. But since the numerator on the right is equal to t_i :

$$\bar{t}_I + w \cong \frac{t_i}{i}.$$

⁴This expression is an important one and might be worthwhile to illustrate with an example. Consider a specific t_i , say the fourth one, t_4 . The values of t_0 , t_1 , t_2 , and t_3 are, obviously, all smaller than t_4 , and t_4 may be expressed in terms of them. Since, in this example, $i = 4$, this is accomplished as follows

$$t_4 = \sum_{j=1}^4 (t_j - t_{j-1}).$$

The expression in parentheses is the difference between adjacent t_j 's. Where $t_0 = 0$, the sum of these differences, up to $j = 4$, is equal to t_4 :

$$\begin{aligned} t_4 &= (t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + (t_4 - t_3) \\ &= (t_1 - t_0 + t_2 - t_1 + t_3 - t_2 + t_4 - t_3) \\ &= t_4 - t_0 \\ &= t_4. \end{aligned}$$

and

$$i(\bar{t}_I + w) \cong t_i. \quad (15)$$

Equation 15 expresses t_i in terms of \bar{t}_I .

The analogous expression for t^* is

$$i(\bar{t}^*_I + w^*) \cong t^*_i. \quad (16)$$

Substituting Equations 15 and 16 into Equation 13 results in

$$A_B(1 - e^{-w^*}) \sum_{i=1}^{\infty} e^{-i(\bar{t}^*_I + w^*)} = \gamma A_B(1 - e^{-w}) \sum_{i=1}^{\infty} e^{-i(\bar{t}_I + w)}. \quad (17)$$

The equation now contains the inverses of the average reinforcement and response rates. In order to solve the equation for rate of responding, $(1/\bar{t}^*_I)$, however, it is necessary to evaluate the infinite sums that appear on both sides of the equation. This may be accomplished by the method of partial sums (see Appendix A for the complete evaluation). The values of the two sums are

$$\sum_{i=1}^{\infty} e^{-i(\bar{t}^*_I + w^*)} = \frac{-e^{-i(\bar{t}^*_I + w^*)}}{e^{-i(\bar{t}^*_I + w^*)} - 1} \quad (18)$$

and

$$\sum_{i=1}^{\infty} e^{-i(\bar{t}_I + w)} = \frac{-e^{-i(\bar{t}_I + w)}}{e^{-i(\bar{t}_I + w)} - 1}. \quad (19)$$

Substituting Equations 18 and 19 into Equation 17 gives

$$A_B(1 - e^{-w^*}) \left[\frac{-e^{-i(\bar{t}^*_I + w^*)}}{e^{-i(\bar{t}^*_I + w^*)} - 1} \right] = \gamma A_B(1 - e^{-w}) \left[\frac{-e^{-i(\bar{t}_I + w)}}{e^{-i(\bar{t}_I + w)} - 1} \right]. \quad (20)$$

This equation only looks complicated. It is still of the general form, $B(s=1) = \gamma R(s=1)$. Notice that the transformed versions of the two signals are perfectly symmetrical. Each consists of three factors: the pulse amplitude, a factor (in parentheses) containing the pulse width, and a factor (in brackets) containing the average interresponse or interreinforcement interval. Equation 20 is less complicated than either Equation 17 or Equation 13 because it contains the desired quantities, viz., \bar{t}_I and \bar{t}^*_I , and has no infinite sums. The equation is, in fact, an ordinary algebraic expression, and solving it for $1/\bar{t}^*_I = R_{out}$ is a routine matter. The algebra will be presented

here for the sake of completeness but one can skip immediately to Equation 22, which is Equation 20 solved for R_{out} , if desired.

Solving Equation 20 for \bar{t}^*_I can be simplified by making the following substitutions:

$$\begin{aligned} a &= A_B(1 - e^{-w}) \\ b &= A_B(1 - e^{-w^*}) \\ N &= \frac{\gamma a}{b} \left[\frac{-e^{-i(\bar{t}_I + w)}}{e^{-i(\bar{t}_I + w)} - 1} \right]. \end{aligned} \quad (21)$$

With these substitutions, Equation 20 becomes

$$\frac{-e^{-i(\bar{t}^*_I + w^*)}}{e^{-i(\bar{t}^*_I + w^*)} - 1} = N.$$

Multiplying through by $e^{-i(\bar{t}^*_I + w^*)} - 1$ and solving for $e^{-i(\bar{t}^*_I + w^*)}$ gives

$$e^{-i(\bar{t}^*_I + w^*)} = \frac{N}{N+1}.$$

Taking the natural logarithm of both sides and solving for \bar{t}^*_I ,

$$\bar{t}^*_I = \ln(1 + N^{-1}) - w^*,$$

and substituting the value of N as given in Equations 21 back into this expression produces

$$\bar{t}^*_I = \ln[1 + (b/\gamma a)(e^{1/\bar{t}_I + w} - 1)] - w^*.$$

Recalling from Equations 5 and 8 that $\bar{t}^*_I = 1/R_{out}$ and $\bar{t}_I = 1/R_{in}$, and inverting both sides gives

$$R_{out} = \{ \ln[1 + (b/\gamma a)(e^{1/R_{in} + w} - 1)] - w^* \}^{-1}.$$

Finally, the complete equation is obtained by performing the remaining resubstitutions as specified by Equations 21:

$$R_{out} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} (e^{1/R_{in} + w} - 1) \right] - w^* \right\}^{-1} \quad (22)$$

Equation 22 is the appropriate form of the transfer function on $R(t)$, and will be referred to as the multivariate rate equation, or simply as the rate equation. It expresses the rate of responding on a VI schedule as a function of the rate of reinforcement (R_{in}), the amplitude and width of the reinforcement pulses (A_R and w), the amplitude and width of the response pulses (A_B and w^*), and the scalar constant, γ , which is characteristic of the system.

THE MULTIVARIATE RATE EQUATION

Equation 22 can be made somewhat more tractable by the following substitutions:

$$1/R_{IN} = 1/R_{in} + w, \quad (23)$$

and

$$1/R_{OUT} = 1/R_{out} + w^*. \quad (24)$$

When w and w^* are negligible, the rates with upper-case subscripts in the above equations are approximations of the rates with lower-case subscripts. In most experiments this simplification is reasonable because response and reinforcer durations are usually brief. Substituting Equations 23 and 24 (i.e., assuming w and w^* are small) into Equation 22 produces

$$R_{OUT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma A_R(1 - e^{-w})} (e^{1/R_{IN}} - 1) \right] \right\}^{-1}. \quad (25)$$

This equation can be made more manageable conceptually by recognizing that it contains two important factors, both of which appear inside the brackets on the right-hand side of the equation. A "rate factor" (the factor involving R_{IN}) is on the extreme right and a "pulse factor" appears to its immediate left. The numerator of the pulse factor is a product that characterizes the response pulses. The denominator of the pulse factor consists of γ and a product that characterizes the reinforcement pulses.

Preliminary Tests of the Rate Equation

To establish a serious claim as a multivariate rate equation for VI performance, Equation 25 must be shown to be at least as good as Herrnstein's hyperbola for univariate cases. Catania and Reynold's (1968) study of VI performance in pigeons and two recent studies of VI performance in human subjects were selected to evaluate the accuracy of the rate equation in describing response rate data generated by variations in rate of reinforcement.

Equation 25 may be written with two free parameters for fitting as follows:

$$R_{OUT} = \left\{ \ln[a_1 e^{1/R_{IN}} + a_0] \right\}^{-1}, \quad (26)$$

where $a_1 = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$ and $a_0 = 1 - a_1$. Notice that the equation predicts that the sum of the two fitting parameters will be unity. A complete derivation of Equation

26 is given in Appendix B. Initial estimates of the fitting constants, a_1 and a_0 , were obtained by linear regression using the method of least squares (Equation 26 is written linearly as $e^{1/R_{OUT}} = a_1 e^{1/R_{IN}} + a_0$). These estimates were then adjusted by iterative methods like those described by Lewis (1960) so that the sum of the squares of the residuals about the curve described by Equation 26 was a minimum. The percentage of data variance accounted for by the equation was calculated according to the method described by deVilliers and Herrnstein (1976) and deVilliers (1977). For the sake of comparison, Herrnstein's hyperbola was fitted to the data with identical procedures.

Catania and Reynolds (1968) studied key pecking reinforced by brief access to mixed grain in six pigeons. For each pigeon grain was presented on up to six different VI schedules that varied reinforcement frequency from 8 to 300 reinforcements per hour. The rate of responding generated at each VI value was determined for all six pigeons. Table 1 shows the percentage of data variance accounted for by the rate equation, the percentage of data variance accounted for by Herrnstein's hyperbola (Equation 1), and the number of data points obtained for each pigeon. Data points were estimated from Catania and Reyn-

Table 1

Percentage of data variance accounted for when response rate varies as a function of reinforcement rate.

Study	No. of points	Percent of variance accounted for	
		Rate equation	Herrnstein's hyperbola
Catania & Reynolds, 1968			
P118	5	89.25	89.91
P121	4	84.59	84.24
P129	5	70.83	70.66
P278	6	94.03	93.30
P279	6	97.98	97.93
P281	4	99.25	99.26
Bradshaw et al., 1976			
BF	5	97.41	96.75
BH	5	97.77	97.19
SM	5	95.50	96.34
AM	5	99.18	99.48
Bradshaw et al., 1977			
BJ	5	96.62	96.08
JL	5	99.83	99.94
VG	5	98.94	98.64

olds' figures. The rate equation describes these data fairly accurately, accounting for between 71% and 99% of the individual pigeons' data variance, with a mean of 89% of the variance accounted for by the equation.

Bradshaw, Szabadi, and Bevan (1976, 1977) essentially replicated Catania and Reynolds' experiment with seven human subjects. They arranged monetary reinforcement (i.e., points that could be exchanged for money) on five different VI schedules for each subject's button pressing. In the first (1976) study, reinforcement rates were varied from 5 to 211 reinforcements per hour and in the second (1977) from 5 to 445 reinforcements per hour. Data points used for fitting were estimated from figures. As shown in Table 1, the rate equation describes these data very accurately, accounting for between 96% and 100% of the individual subject's data variance, with a mean of 98%

of the variance accounted for by the equation. Figure 3 shows plots of the rate equation fitted to data from four of Bradshaw et al.'s subjects.

The sum of the fitting constants for the rate equation was 1.0 (rounded to the first decimal place; see Appendix B) for every subject listed in Table 1, as predicted by the equation.

In general, the rate equation provides as good a description of these data as does Herrnstein's hyperbola. Both equations account for an average of 94% of the data variance. These fits also show that the rate equation is hyperbola-like. Since the relationship between response rate and amount or immediacy of reinforcement is also hyperbolic (deVilliers and Herrnstein, 1976), the rate equation will provide a reasonably good fit to these univariate cases as well.

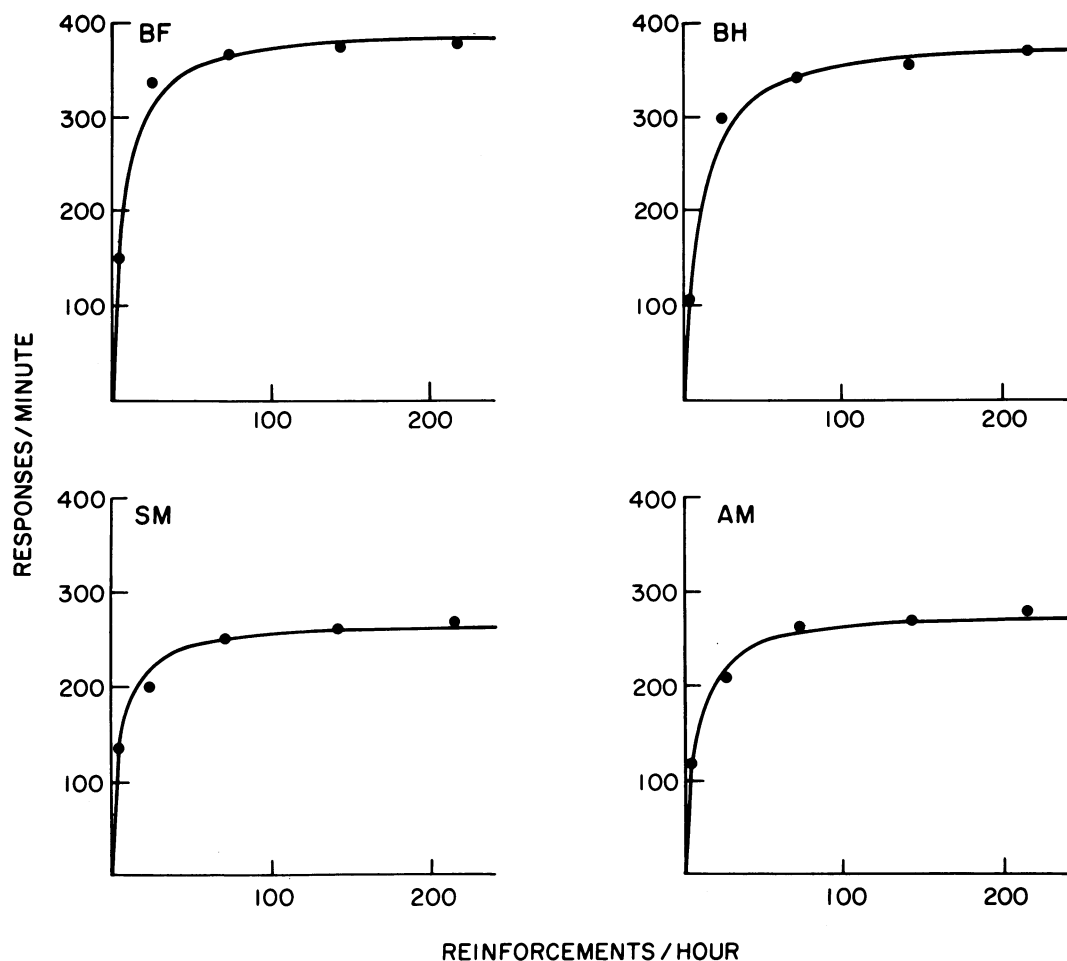


Fig. 3. Rate equation fitted to data from Bradshaw et al.'s (1976) human subjects.

The only quantity that is properly termed a parameter in Equation 25 is γ . The value of γ characterizes the system under study and must be extracted statistically or solved for analytically in cases where the differential equation describing the system is unknown. The quantities A_B , w^* , A_R and w , on the other hand, characterize the input and output of the system, are variable (i.e., experimentally manipulable), and are presumably susceptible to direct measurement (see next section). Unfortunately, the structure of the fitting constants of Equation 26 do not permit a determination of the value of γ . For Catania and Reynolds' six pigeons, the values of $a_1 = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$ range from .05 to 2.63, with a median value of .12. The high value is from P121 and is extremely deviant since a_1 is less than .16 for all the other pigeons. For Bradshaw, et al.'s human subjects a_1 ranges from .01 to .12 with a median value of .04, which is an order of magnitude less than the median value for the pigeons. The most that can be concluded from these figures is that the product $\gamma A_R(1 - e^{-w})$ is an order of magnitude bigger than the product $A_B(1 - e^{-w^*})$ for most of Catania and Reynolds' pigeons, and is two orders of magnitude bigger than $A_B(1 - e^{-w^*})$ for most of Bradshaw et al.'s human subjects. This says nothing, of course, about the relative sizes of the quantities that appear in these products.

Although Equation 25 is not a rectangular hyperbola like Herrnstein's equation, it yields a rectangular hyperbola of the form

$$y = \frac{x}{a + bx},$$

in the variates $e^{-1/R_{IN}}$ and $e^{-1/R_{OUT}}$:

$$e^{-1/R_{OUT}} = \frac{e^{-1/R_{IN}}}{a + be^{-1/R_{IN}}},$$

where $a = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$ and $b = 1 - a$. Rearranging this equation produces a hyperbola in Herrnstein's form:

$$e^{-1/R_{OUT}} = \frac{b \cdot e^{-1/R_{IN}}}{e^{-1/R_{IN}} + ab^{-1}}.$$

(The complete derivation of both equations is given in Appendix C.) Thus Equation 25 not only accounts for certain univariate cases as well as Herrnstein's equation, but it also yields a form of Herrnstein's equation.

Finally, Equation 25 predicts a form of ordinary matching in concurrent situations:

$$\frac{e^{1/R_{1OUT}} - 1}{e^{1/R_{2OUT}} - 1} = \frac{e^{1/R_{1IN}} - 1}{e^{1/R_{2IN}} - 1},$$

where the numerical subscripts refer to the two response alternatives. The rate equation also predicts a form of proportional ratio matching (deVilliers, 1977):

$$\frac{e^{1/R_{1OUT}} - 1}{e^{1/R_{2OUT}} - 1} = \left[\frac{a_1}{a_2} \right] \frac{e^{1/R_{1IN}} - 1}{e^{1/R_{2IN}} - 1},$$

where $a_1 = [A_{1B}(1 - e^{-w^*})]/[A_{1R}(1 - e^{-w_1})]$ and $a_2 = [A_{2B}(1 - e^{-w^*})]/[A_{2R}(1 - e^{-w_2})]$. Proportional ratio matching is the more general statement of the matching law since it accounts for deviations from matching that are commonly found in concurrent situations (Baum & Rachlin, 1969). The complete derivation of both forms of matching is given in Appendix C.

Application to Multivariate Cases

Equation 25 is reproduced here for convenience:

$$R_{OUT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma A_R(1 - e^{-w})} (e^{1/R_{IN}} - 1) \right] \right\}^{-1}. \quad (25)$$

In this equation, a reinforcement pulse is characterized by the product $A_R(1 - e^{-w})$, where A_R is a point along the value-like dimension, $R(t)$, and w is the pulse width. Reinforcement pulse amplitude and width may be consolidated into a single quantity, reinforcer "power" (cf. Herrnstein, 1971), as follows:⁵

$$P_R \equiv A_R(1 - e^{-w}).$$

Making this substitution in Equation 25 gives

$$R_{OUT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma P_R} (e^{1/R_{IN}} - 1) \right] \right\}^{-1}. \quad (27)$$

Reinforcer power is conceptually similar to reinforcer value. In Equation 27, P_R absorbs all reinforcer characteristics, other than rate of delivery, including the value-like amplitude parameter. This comprehensiveness permits direct measurement of P_R , which is not possible in the case of A_R . The value of P_R may be determined in a concurrent VI VI schedule where relative response rate matching is as-

⁵Notice that when w is small, $(1 - e^{-w}) \cong w$, and $P_R \cong A_R w$. That is, when w is small, P_R is approximately equal to the area of a reinforcement pulse.

sumed to hold. When matching is assumed and reinforcement rates are held constant across response alternatives, deviations from matching generated by reinforcers that differ in amount, immediacy, type, or in any other characteristic, may be used to scale the power of the reinforcers studied. The scaled reinforcer values can be taken as determinations of P_R . Several authors have discussed this type of matching-based reinforcer scaling (Baum, 1974; Herrnstein, 1971; Miller, 1976).

A test of Equation 27 is possible on data collected by Miller (1976). He studied the key pecking of four pigeons on concurrent VI VI schedules of reinforcement. Reinforcement consisted of three types of grain, viz., buckwheat, hemp, and wheat, each of which was delivered on five different component VI schedules. Variable-interval values were selected so as to vary reinforcement rate from approximately 10 to 50 reinforcements per hour. Each pigeon was exposed to all combinations (15) of grain type and reinforcement rate. Based on deviations from matching, Miller scaled the "quality" or power of the three types of grain, choosing buckwheat as a standard, equal to 10 units (dimension unspecified) of reinforcer power.

Since Miller determined the response rate generated by each type of grain at all five rates of reinforcement, Equation 27, where R_{OUT} is expressed as a joint function of P_R and R_{IN} , can be fitted to the data by multiple regression techniques. Equation 27 is written for fitting as follows:

$$\ln(e^{1/R_{OUT}} - 1) = a_2 \ln(1/P_R) + a_3 \ln(e^{1/R_{IN}} - 1) + a_0 \quad (28)$$

where $a_2 = a_3 = 1$, and $a_0 = \ln\{[A_B(1 - e^{-w^*})]/\gamma\}$. A complete derivation of Equation 28 is given in Appendix B. Notice that the equation predicts that two of the regression constants (a_2 and a_3) will equal unity.⁶ Figure 4 shows fits of Equation 28 to data from the individual birds and to data averaged across all birds. The coordinate axes have been rotated 180° for convenience. The top figure in the lower right of each graph is the square of the multiple regression coefficient, R , which estimates

the proportion of data variance accounted for by the equation. The values of the fitting constants, a_2 and a_3 , are also given for each function. Except for Pigeon 254, these values are very close to 1, as predicted by the rate equation.

A more stringent test of Equation 27 is possible by restricting the regression to a single variate given by the algebraic combination of R_{IN} and P_R as specified by the equation. Thus, Equation 27 can be written as

$$e^{1/R_{OUT}} = a_1 \left[\frac{e^{1/R_{IN}} - 1}{P_R} \right] + a_0 \quad (29)$$

where $a_1 = [A_B(1 - e^{-w^*})]/\gamma$ and $a_0 = 1$. A complete derivation of Equation 29 is given in Appendix B. Notice that the equation predicts that a_0 will equal unity (see footnote 6). Table 2 shows the results of ordinary linear regression by the method of least squares. The coefficients of determination (product-moment correlation coefficient squared) show that, even for this more stringent test, Equation 29 accounts for a large proportion of the data variance. In addition, the value of a_0 is 1 for all birds, as predicted by the equation.

As was the case for the univariate fits, the structure of the fitting constants for these multivariate fits do not permit estimates of γ , nor do they provide information about the relative sizes of the other quantities that appear in the constants.

DISCUSSION

It is an interesting novelty in behavioral science to find an a priori mathematics that produces a useful empirical equation. In addition, the simplicity of a rate equation that is written in two reinforcement variables is appealing. The two variables, power and rate of delivery, constitute a complete description of reinforcement. Power is readily determined

Table 2

Results of linear regression restricted to the single variate, $(e^{1/R_{IN}} - 1)/P_R$

Subject	r ²
P254	.95
P255	.96
P452	.97
P43	.91
Average function	.99

Note. Data on which table is based are from Miller (1976). There are fifteen data points per fit.

⁶This equation must be fit as written for the same reason that Equation 26 had to be fit with two instead of one free parameter, viz., because $w \neq w^* \neq 0$ for any real behavior or reinforcement. See explanation in Appendix B.

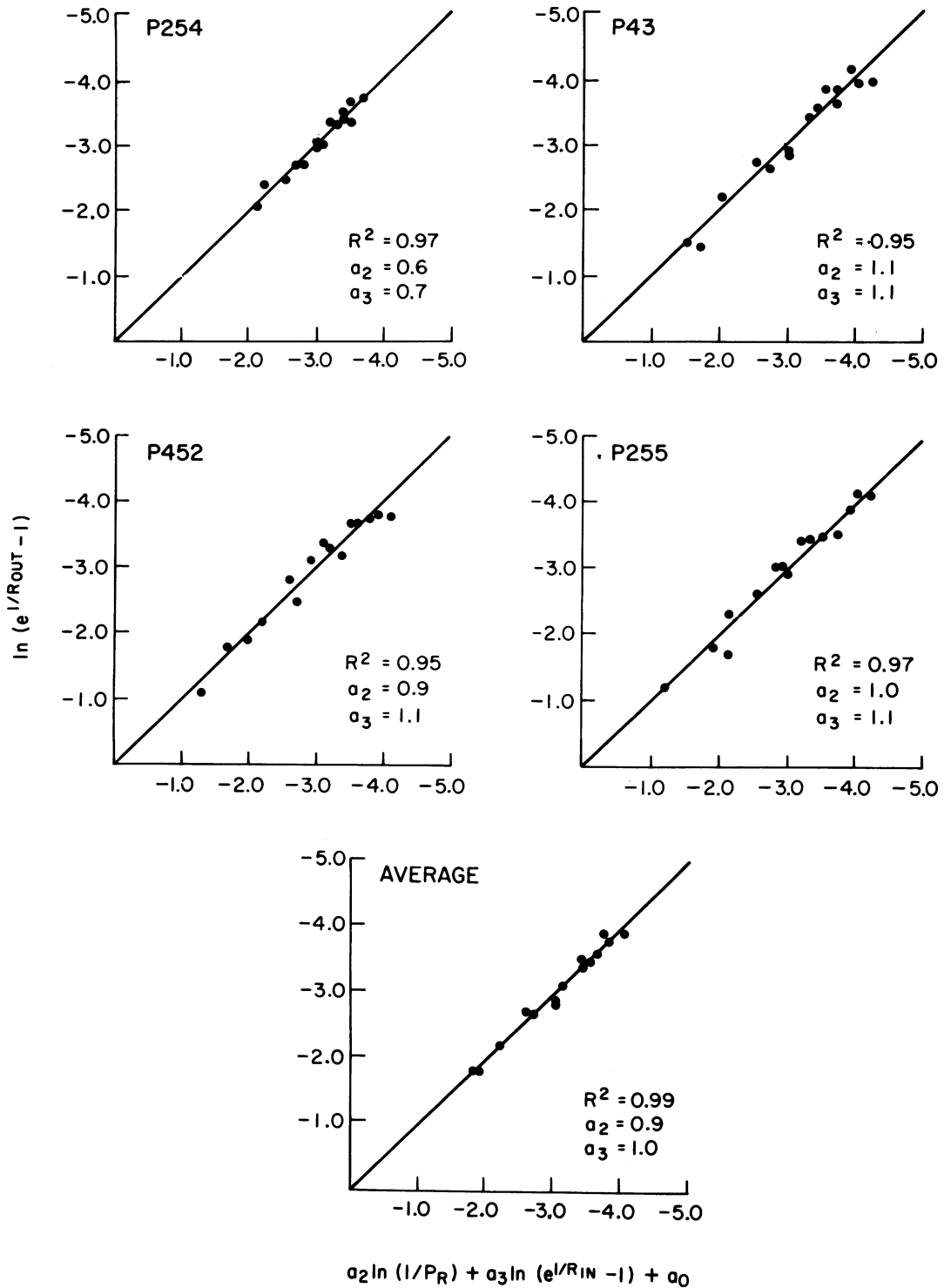


Fig. 4. Rate equation fitted to data from Miller's (1976) pigeons where rate and power of reinforcement were varied simultaneously. The abscissae and ordinates show predicted and obtained values respectively. The square of the multiple regression coefficient (R) is given for each fit.

in a concurrent situation where matching is assumed to hold.

It is important to recognize that there are two distinct steps in the calculation of the multivariate rate equation. First, $R(t)$ and $B(t)$ are written in explicit mathematical form, and second, a transfer function is written between them. Each of these steps requires a separate set of assumptions. For different inputs (i.e., for schedules other than VI), the assumptions required to write $R(t)$ and $B(t)$ may change, but the single assumption required to write the transfer function, viz., that the system is linear, will always be the same. Although this latter assumption is not very restrictive, the form of the transfer function in behavioral applications depends not only on the assumption of linearity but also on the way in which $R(t)$ and $B(t)$ are written. The assumptions necessary for writing $R(t)$ and $B(t)$ restrict the mathematics and result in a transfer function with properties that are not required by the mere assumption of linearity. In the VI case, for example, a rate equation where response rate is directly proportional to reinforcement rate (i.e., $R_{OUT} = kR_{IN}$) does not contradict the assumption of linearity. But when the additional restrictions required by writing $R(t)$ and $B(t)$ as in Equations 4 and 7 are added (recall, for example, the third assumption which implies an upper limit on the rates of reinforcement and responding), this direct proportionality does not hold.

It is also important to recognize that the modification of the Laplace transform method that is required for behavioral applications means that any equation derived by the method holds only for the input that has entered into the calculation. In the present case, this means that the multivariate rate equation holds only for VI inputs. It should be possible, however, to calculate transfer functions for other simple schedule inputs (or, indeed, for any input) by writing the $R(t)$'s and $B(t)$'s properly.

A noteworthy feature of Equation 27 is the appearance of the product $A_R(1 - e^{-w*})$. This "behavioral" product is analogous to the "reinforcement" product, $A_R(1 - e^{-w}) \equiv P_R$. It is possible that a symmetric scaling procedure for response forms will produce values analogous to P_R that may be substituted into the rate equation. If so, only one free parameter will remain in Equation 27, viz., γ , and

its value may be estimated by statistical methods.

Research directions with respect to the multivariate rate equation are fairly clear. Further tests of simultaneous variations of P_R and R_{IN} are necessary, particularly in view of the fact that in Miller's experiment, both quantities were varied over relatively small ranges. The almost unexplored area of response form scaling, and its relationship to Equation 27 needs to be studied, and the forms of matching predicted by the equation require testing.

APPENDIX A

Evaluation of

$$R(s) = \sum_{i=1}^{\infty} \int_{t_i}^{t_i+w} A_R e^{-st} dt$$

Since A_R is a constant, it may be removed from both the integral and the infinite sum:

$$R(s) = A_R \sum_{i=1}^{\infty} \int_{t_i}^{t_i+w} e^{-st} dt.$$

The integral in the above expression is an elementary form and is not difficult to evaluate. Since $\int e^{ax} dx = e^{ax}/a$,

$$R(s) = A_R \sum_{i=1}^{\infty} -e^{-st}/s \Big|_{t_i}^{t_i+w}.$$

Removing the constant, s ,

$$R(s) = (A_R/s) \sum_{i=1}^{\infty} -e^{-st} \Big|_{t_i}^{t_i+w},$$

and substituting in the limits of integration gives

$$R(s) = (A_R/s) \sum_{i=1}^{\infty} (-e^{-s(t_i+w)} + e^{-st_i})$$

or

$$R(s) = (A_R/s) \sum_{i=1}^{\infty} (e^{-st_i} - e^{-st_i-sw}).$$

Factoring,

$$R(s) = (A_R/s) \sum_{i=1}^{\infty} e^{-st_i} (1 - e^{-sw})$$

and removing the constant factor, $(1 - e^{-sw})$, gives the transformed version of $R(t)$:

$$R(s) = (A_B/s) (1 - e^{-s}) \sum_{i=1}^{\infty} e^{-s^i}. \quad (10)$$

Evaluation of Infinite Sums

Consider the k^{th} partial sum, S_k , of the infinite sum appearing on the right hand side of Equation 17:

$$S_k = e^{-(\bar{t}_I + w)} + e^{-(\bar{t}_I + w)^2} + \dots + e^{-(\bar{t}_I + w)^k}.$$

Multiplying both sides by $e^{-(\bar{t}_I + w)}$ gives

$$e^{-(\bar{t}_I + w)} S_k = e^{-(\bar{t}_I + w)^2} + e^{-(\bar{t}_I + w)^3} + \dots + e^{-(\bar{t}_I + w)^k} + e^{-(\bar{t}_I + w)^{(k+1)}}.$$

Notice that the sum on the right is the $(k+1)^{\text{th}}$ partial sum minus the quantity $e^{-(\bar{t}_I + w)}$. Thus

$$e^{-(\bar{t}_I + w)} S_k = S_{k+1} - e^{-(\bar{t}_I + w)}.$$

In the lim $S_k = S_{k+1}$ since the series is convergent. In other words, given any $\epsilon > 0$, a k may be found such that

$$|S_{k+1} - S_k| < \epsilon, \text{ all } h > k.$$

Accordingly, in the limit,

$$e^{-(\bar{t}_I + w)} S_{\infty} = S_{\infty} - e^{-(\bar{t}_I + w)},$$

and

$$e^{-(\bar{t}_I + w)} S_{\infty} - S_{\infty} = -e^{-(\bar{t}_I + w)}.$$

$$S_{\infty} [e^{-(\bar{t}_I + w)} - 1] = -e^{-(\bar{t}_I + w)}.$$

Solving for S_{∞} and stating the equation properly:

$$\lim_{k \rightarrow \infty} S_k = \frac{-e^{-(\bar{t}_I + w)}}{e^{-(\bar{t}_I + w)} - 1}.$$

Finally, since the limit of the partial sums is the value of the infinite sum,

$$\sum_{i=1}^{\infty} e^{-(\bar{t}_I + w)^i} = \frac{-e^{-(\bar{t}_I + w)}}{e^{-(\bar{t}_I + w)} - 1}.$$

The infinite sum on the left hand side of Equation 17 is evaluated in an identical fashion.

APPENDIX B

Derivation of Equation 26

From Equation 25,

$$R_{OVT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} (e^{1/R_{IN}} - 1) \right] \right\}^{-1},$$

$$e^{1/R_{OVT}} = \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} (e^{1/R_{IN}} - 1) + 1,$$

$$e^{1/R_{OVT}} = \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} e^{1/R_{IN}} + \left[1 - \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} \right].$$

Let $a_1 = [A_B(1 - e^{-w^*})]/[\gamma A_B(1 - e^{-w})]$ and $a_0 = 1 - a_1$; then

$$e^{1/R_{OVT}} = a_1 e^{1/R_{IN}} + a_0,$$

and

$$R_{OVT} = \left\{ \ln [a_1 e^{1/R_{IN}} + a_0] \right\}^{-1}. \quad (26)$$

It may seem that since Equation 26 has only one free parameter, it should be fit with a_0 fixed at $1 - a_1$:

$$R_{OVT} = \left\{ \ln [a_1 e^{1/R_{IN}} + (1 - a_1)] \right\}^{-1}. \quad (B1)$$

But it is important to recognize that Equations 23 through 26 constitute a simplification that is possible only when w and w^* are assumed to be negligible. When this is not the case, Equation B1 should be written

$$R_{out} = \left\{ \ln [(a_1 e^{w^*} e^{-w}) e^{1/R_{IN}} + (1 - a_1) e^{-w^*}] \right\}^{-1}. \quad (B2)$$

This equation is readily obtained from Equation 22 by letting $[A_B(1 - e^{-w^*})]/[\gamma A_B(1 - e^{-w})] = a_1$. Inverting, adding w^* to, and exponentiating both sides, and rearranging terms, gives

$$e^{1/R_{out} + w^*} = a_1 e^{1/R_{IN} + w} + (1 - a_1).$$

Multiplying both sides of this equation by e^{-w^*} , taking the natural logarithm, and inverting, produces Equation B2. Letting $a_1' = a_1 e^{w^*} e^{-w}$ and $a_0' = (1 - a_1) e^{-w^*}$, Equation B2 may be written with two free parameters for fitting as

$$R_{out} = \left\{ \ln [a_1' e^{1/R_{IN}} + a_0'] \right\}^{-1}.$$

It is clear that this equation must be fit with two free parameters since $a_0' \neq 1 - a_1'$. But notice that Equation B2 (two free parameters) reduces to Equation B1 (one free parameter) only when $w = w^* = 0$. For real behavior and real reinforcement, w and w^* may be negligible, but they can never equal zero. Hence, the statement $a_0 = 1 - a_1$ is never strictly true, and Equation B1 must be fit with two free parameters. If the assumption that w and w^* are negligible is reasonable, then the sum of the two free parameters will be approximately equal to unity.

Derivation of Equation 28

From Equation 27,

$$R_{OUT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma P_B} (e^{1/R_{IN}} - 1) \right] \right\}^{-1},$$

$$e^{1/R_{OUT}} - 1 = \frac{A_B(1 - e^{-w^*})}{\gamma P_B} (e^{1/R_{IN}} - 1),$$

$$\ln(e^{1/R_{OUT}} - 1) = \ln(1/P_B) + \ln(e^{1/R_{IN}} - 1) + \ln \left[\frac{A_B(1 - e^{-w^*})}{\gamma} \right].$$

And

$$\ln(e^{1/R_{OUT}} - 1) = a_2 \ln(1/P_B) + a_2 \ln(e^{1/R_{IN}} - 1) + a_0, \quad (28)$$

where $a_2 = a_3 = 1$, and $a_0 = \ln\{[A_B(1 - e^{-w^*})]/\gamma\}$.

Derivation of Equation 29

From Equation 27,

$$R_{OUT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma P_B} (e^{1/R_{IN}} - 1) \right] \right\}^{-1},$$

$$e^{1/R_{OUT}} = \frac{A_B(1 - e^{-w^*})}{\gamma} \left[\frac{e^{1/R_{IN}} - 1}{P_B} \right] + 1.$$

Hence

$$e^{1/R_{OUT}} = a_1 \left[\frac{e^{1/R_{IN}} - 1}{P_B} \right] + a_0, \quad (29)$$

where $a_1 = [A_B(1 - e^{-w^*})]/\gamma$ and $a_0 = 1$.

APPENDIX C

Derivation of Hyperbolic Forms

From Equation 25, as in Appendix B,

$$R_{OUT} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} (e^{1/R_{IN}} - 1) \right] \right\}^{-1},$$

$$e^{1/R_{OUT}} = \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} e^{1/R_{IN}} + \left[1 - \frac{A_B(1 - e^{-w^*})}{\gamma A_B(1 - e^{-w})} \right].$$

Let $a = [A_B(1 - e^{-w^*})]/[\gamma A_B(1 - e^{-w})]$ and $b = 1 - a$; then

$$e^{1/R_{OUT}} = ae^{1/R_{IN}} + b,$$

$$\frac{1}{e^{-1/R_{OUT}}} = \frac{a}{e^{-1/R_{IN}}} + \frac{be^{-1/R_{IN}}}{e^{-1/R_{IN}}},$$

$$\frac{1}{e^{-1/R_{OUT}}} = \frac{a + be^{-1/R_{IN}}}{e^{-1/R_{IN}}},$$

$$e^{-1/R_{OUT}} = \frac{e^{-1/R_{IN}}}{a + be^{-1/R_{IN}}}.$$

Herrnstein's form is produced by multiplying the right hand side of this hyperbola by b^{-1}/b^{-1}

$$e^{-1/R_{OUT}} = \frac{b^{-1} e^{-1/R_{IN}}}{e^{-1/R_{IN}} + ab^{-1}}.$$

Derivation of Forms of Matching

From Equation 25,

$$e^{1/R_{1OUT}} - 1 = \frac{A_{1B}(1 - e^{-w^*})}{\gamma A_{1B}(1 - e^{-w_1})} (e^{1/R_{1IN}} - 1),$$

and

$$e^{1/R_{2OUT}} - 1 = \frac{A_{2B}(1 - e^{-w^*})}{\gamma A_{2B}(1 - e^{-w_2})} (e^{1/R_{2IN}} - 1),$$

The numerical subscripts (1 and 2) refer to the two response alternatives. Let $a_1 = [A_{1B}(1 - e^{-w^*})]/[A_{1B}(1 - e^{-w_1})]$ and $a_2 = [A_{2B}(1 - e^{-w^*})]/[A_{2B}(1 - e^{-w_2})]$; then

$$e^{1/R_{1OUT}} - 1 = \frac{a_1}{\gamma} (e^{1/R_{1IN}} - 1),$$

and

$$e^{1/R_{2OUT}} - 1 = \frac{a_2}{\gamma} (e^{1/R_{2IN}} - 1).$$

Dividing these two equations gives a form of proportional ratio matching

$$\frac{e^{1/R_{1OUT}} - 1}{e^{1/R_{2OUT}} - 1} = \left[\frac{a_1}{a_2} \right] \frac{e^{1/R_{1IN}} - 1}{e^{1/R_{2IN}} - 1}.$$

When $a_1 = a_2$, that is, when the reinforcement spikes for the two alternatives are identical, and the response spikes are also identical, a form of relative proportion matching is predicted:

$$\frac{e^{1/R_{1OUT}} - 1}{e^{1/R_{2OUT}} - 1} = \frac{e^{1/R_{1IN}} - 1}{e^{1/R_{2IN}} - 1}.$$

APPENDIX D

The calculation of a transfer function can be illustrated with a simple example. Consider the following differential equation of the first order:

$$F_i(t) = F_o(t) + k \frac{dF_o(t)}{dt},$$

where $F_i(t)$ is some input function, $F_o(t)$ is the resulting output function, and $F_o(t=0) = 0$, i.e., the output at time $t=0$ is 0. The coefficient, k , is a constant property of the linear system.

Taking the Laplace transform of both sides of this equation gives

$$\int_0^{\infty} F_i(t)e^{-st}dt = \int_0^{\infty} F_o(t)e^{-st}dt + k \int_0^{\infty} \frac{dF_o(t)}{dt} e^{-st}dt.$$

Evidently,

$$F_i(s) = F_o(s) + k \int_0^{\infty} \frac{dF_o(t)}{dt} e^{-st}dt.$$

A table of Laplace transforms shows that the transform of $dF(t)/dt$ is $sF(s) - F(t=0)$. Thus, recalling that $F_o(t=0) = 0$, the remaining integral is equal to $sF_o(s)$, and the complete transformed equation is written

$$F_i(s) = F_o(s) + ksF_o(s).$$

Factoring,

$$F_i(s) = F_o(s)(1 + ks),$$

and calculating the transfer function, i.e., the ratio of transformed output to transformed input, gives

$$\frac{F_o(s)}{F_i(s)} = \frac{1}{1 + ks} = G(s).$$

Thus

$$F_o(s) = G(s)F_i(s)$$

where, for this linear system, $G(s) = 1/(1 + ks)$.

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